Dynamic of a Rolling Golf Ball

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Contents

1 Energy

At the beginning of a put, the ball's kinetic energy is E_0; when the ball comes to rest (after a distance S):

$$E = E_{diss} + E_{pot} \tag{1}$$

Where:

- E_{diss}: "lost" (dissipated) energy due to rolling friction.
- E_{pot}: Potential energy caused by height difference.

1.1 Kinetic Energy of a rolling Golf Ball

The kinetic energy consist of *translation* E_{trans} and *rotation* E_{rot}.

$$E_{kin} = E_{trans} + E_{rot}$$

Furthermore:

$$E_{trans} = \frac{m}{2}v^2 \tag{2}$$

$$E_{rot} = \frac{J}{2}\omega^2 \tag{3}$$

At the condition of pure rolling (without gliding):

$$v = r \cdot \omega \tag{4}$$

Where:

- m: Mass of the golf ball.
- v: Velocity of the golf ball.
- J: Moment of inertia of the golf ball.
- ω: Angular velocity of the golf ball.
- r: Radius of the golf ball.

Under the (simplifying) assumption of homogeneous mass distribution:

$$J = \frac{2}{5}mr^2 \tag{5}$$

From 2, 3, 5 and 4 it follows:

$$E_{kin} = \frac{m}{2}v^2\left(\frac{7}{5}\right) \tag{6}$$

1.2 Dissipation and Potential Energy

The work done by rolling friction is:

$$E_{diss} = \int_0^S mgf \cdot dx \tag{7}$$

Where f is the coefficient of rolling friction. The force due to the friction $R = fmg$ *acts against the direction of motion.*

$$\vec{R} = fmg\frac{\vec{v}}{|\vec{v}|} \tag{8}$$

Under the assumption of a constant inclination (angle θ) 7 becomes:

$$E_{diss} = m \cdot g \cdot f \cdot \cos(\theta) \cdot S \tag{9}$$

The potential energy at the and of the path is:

$$E_{pot} = m \cdot g \cdot h \tag{10}$$

Where:

- m: Mass of the golf ball.
- g: Acceleration of gravity on the surface of the earth ($9.81 m/s^2$).
- f: Coefficient of rolling friction).
- h: Height difference.

So, (1) becomes:

$$m \cdot g \cdot h + m \cdot g \cdot f \cdot \cos\theta \cdot S = \frac{m}{2} v^2 \left(\frac{7}{5}\right) \tag{11}$$

2 Equation of motion of a rolling ball

2.1 One-dimensional motion

The system can be described by a coordinate x. By denoting $v = \frac{dx}{dt} = \dot{x}$ and using (6) und (10), the Lagrange function becomes:

$$L = T - U = \frac{7}{10} m\dot{x}^2 - mgx\sin(\theta)$$

with

$$\frac{d}{dt}\frac{dL}{d\dot{x}} - \frac{dL}{dx} = 0$$

it follows:

$$m\ddot{x} = -\frac{5}{7} mg\sin(\theta) \tag{12}$$

In (12) the rolling friction f_r(see (8)) has to be added. Finally, one obtains:

$$\ddot{x} = -\frac{5}{7} g(\sin(\theta) - f_r\cos(\theta)) \tag{13}$$

2.2 Rolling Ball on an Inclined Plane

Without restriction of generality, the y-axis is chosen to be parallel to the gradient (Fig. 1).

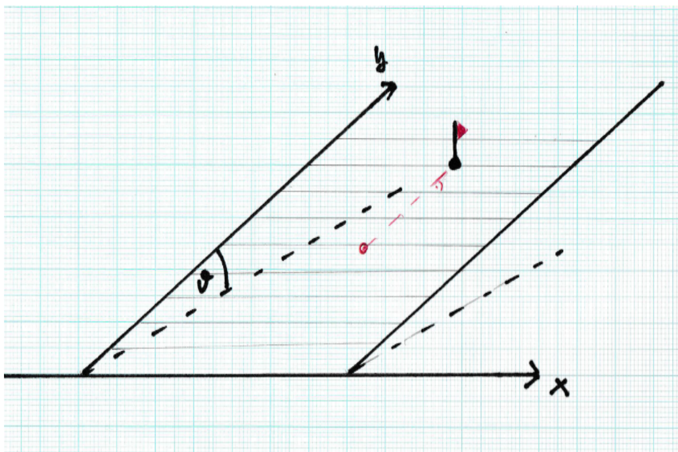

Figure 1: Fig. Inclined plane: Direction of axes.

A point P on the plane is represented by the vector $\vec{\mathbf{x}} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Since the friction is a vector pointing in the (negative) direction of the motion, it follows:

$$\vec{R} = fmg\frac{\vec{v}}{|\vec{v}|}$$

$$\vec{R} \parallel \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \tag{14}$$

The direction of motion can also be expressed by an angle ϕ with:

$$\tan(\phi) = \frac{dy}{dx} \tag{15}$$

As the inclination is less than 4-5%, we will make the approximations:

$$\sin(\theta) \approxeq \theta \quad \text{and} \quad \cos(\theta) \approxeq 1$$

Lagrange formalism for the derivation of the equation of motion

$$L = T - U = \frac{7}{5}m(\dot{x}^2 + \dot{y}^2) - mgy\sin(\theta) = \frac{7}{5}m(\dot{x}^2 + \dot{y}^2) - mgy \cdot \theta$$

one finally obtains as in (13):

$$\ddot{\mathbf{x}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\frac{5}{7}\frac{fg}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{5g\theta}{7} \end{pmatrix} \tag{16}$$

or

$$\ddot{\mathbf{x}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\frac{5}{7}g \cdot \begin{pmatrix} f\cos(\phi) \\ f\sin(\phi) + \theta \end{pmatrix}; \quad \phi = \arctan(\frac{\dot{y}}{\dot{x}}) \tag{17}$$

The equations (16, 17) are a system of two coupled differential equations, for which there is no analytical solution in closed form.

3 Input data for a put

3.1 Known values

A player knows the following data for the put to be executed:

Table 1: Parametrization of a put (pure rolling)

	Symbol
Speed of the green	D_s
Inclination of the green	θ
Distance ball to hole	S
Angle between fall-line and line ball-hole	α
Speed at target	$v_e(rps)$

The coefficient of rolling friction can be derived from the Stimp meter distance S by:

$$f = \frac{7v_0^2}{10 \cdot D_s \cdot g} \tag{18}$$

where v_0 is the velocity a golf ball when it leaves the Stimp meter (1.83 m/s) .

3.2 Parameters for the execution of the put

For a successful put, the following values are needed:

1. The angle φ the ball starts its trajectory (or equivalently $\Delta\varphi$, the difference between initial direction and the line ball-hole).

2. The length S_{corr}, i.e, the length if it were on a plane surface with the same green speed D_s.

S_{corr} ad $\Delta\varphi$ are calculated from the initial velocities $v_{x;0}$ and $v_{y;0}$ a ball had to be given, *if it had a pure rolling motion from the beginning.*

Alternative dimensionless form of the equation of motion As it is shown by Grober (see Grober, 2011), that by rescaling the time, the equation of motion depends only on the product $D_s \times \theta$.
I.e. the initial velocities $v_{x;0}$ and $v_{y;0}$ are the same, when the problem ist standardized:

$$D_s \to \lambda \cdot D_s; \quad S \to \lambda \cdot S; \quad \theta \to \theta/\lambda$$

Example For a put of length $S = 2.5$ m with green velocity $D_s = 10$ ft, inclination 2%, angle to the fall line $\alpha = 57^\circ$ and desired velocity at the target (hole) of 3 rps (= 0.402 m/s) the initial velocities are:

$$v_{x;0} = 1.442 \text{ m/s} \quad \text{and} \quad v_{y;0} = 1.109 \text{ m/s}$$

These initial velocities ar the same as for a put of length $S = 2.25$ m with green velocity $D_s = 9$ ft, inclination 2.22% and same angle to the fall line α and same desired velocity at the target (in this example, $\lambda = 0.9$).

This will be extremely useful for the calculation of the put trajectory!

4 Numerical solution of the equation of motion

4.1 Calculation of the trajectory

The problem can be solved with a numeric ODE solver by.

1. Start at the target with *initial conditions = terminal conditions* $|\vec{v_0}| = |\vec{v_e}|$.
2. $v_{x;0} = |\vec{v_0}| \cos(\gamma); \quad v_{y;0} = |\vec{v_0}| \sin(\gamma)$ (see Fig. 2).
3. *Let the time run backwards.*
4. Find an angle γ, such that the trajectory (running backwards!) passes close (distance $\leq \epsilon$) to the original lie of the ball $P_0 = (x_0, y_0)$.
5. Extract $v_x(P_0)$ and $v_y(P_0)$.

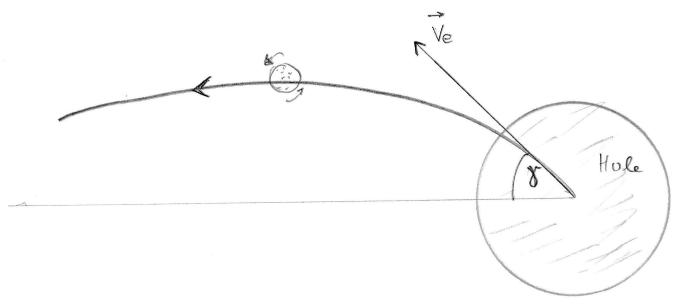

Figure 2: Fig. 2 Initial angle γ when running the time backwards.

The hole procedure has to be embedded in an optimizing algorithm in order to find the angle γ.

4.2 Calculation of the initial speed and angle

By means of the velocities $v_x(P_0)$ and $v_y(P_0)$ the put direction can be calculated as follows:
Initial direction:

$$\varphi = -\arctan(v_y(P_0)/v_x(P_0)) \tag{19}$$

Initial speed:

$$v_0 = \sqrt{v_x(P_0)^2 + v_y(P_0)^2} \tag{20}$$

The initial speed v_0 corresponds to the speed of a put on a plane surface with the same green peed D_s of length

$$S_0 = \frac{7v_0^2}{10 \cdot g \cdot f} \tag{21}$$

Remark S_0 is the distance, where the ball would stop; at the hole, it would have the velocity v_e.

5 Approximation by a Gaussian Process Model

The procedure described in the last section is computationally intensive and probably too slow for an app in mobile phone. Therefore, I propose an approximation by a so-called Gaussian-Process Model as described for example in (Abt, 1998) or in (MacDonald et al., 2015). Basically, it is like an interpolation algorithm;

1. On a grid of parameters α, $D_s \cdot \theta$ and S, the exact values φ and v_0 are calculated by numerically solving the ODE (16, 17).
2. Gaussian Process Model needs the inputs X and outputs Y from the grid with the exact solutions.
3. The result of the Gaussian Process Model is:
   - Coefficients $(\beta_1, \beta_2, \beta_3)$.
   - Gaussian correlation matrix R.
4. New put parameters $(\alpha_{new}, S_{new}, D_s\theta_{new})$
5. With X, Y and R, a the estimated values $\hat{\varphi}(\alpha_{new}, S_{new}, D_s\theta_{new})$ and$\hat{v_0}(\alpha_{new}, S_{new}, D_s\theta_{new})$.
6. Only matrix multiplication (!large matrices) are used for the estimation of $\hat{\varphi}$ and $\hat{v_0}$.

There is a model for φ and v_0 for each distinctive value of v_e : 2 rps, ..., 6 rps.

References

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MacDonald, B., Ranjan, P. and Chipman, H. (2015). Gpfit: An r package for fitting a gaussian process model to deterministic simulator outputs. *Journal of Statistical Software* **64** 123.
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